M434/Specimen

Fourth Level Course Examination

Differential Geometry

Time allowed: 3 hours

There are **TWO** parts to this examination. You should attempt **BOTH** parts. You may attempt **ALL** the questions in Part I and you should answer not more than **THREE** questions from Part II.

You are advised to spend 90 minutes on Part I and 80 minutes on Part II leaving yourself about 10 minutes for checking. Part I carries 55% of the total marks while Part II carries 45% of the total marks.

You are advised to show all your working and to give reasons for all your answers. Begin each answer on a new page of the answer book.

At the end of the examination

Check that you have written your name, examination number and personal identifier on each answer book used. **Failure to do so may mean that your paper cannot be identified.**

Write the numbers of the questions that you have attempted in the spaces provided on the front of the answer book.

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1.1

PART I

You may attempt **ALL** *the questions in this part and are advised to spend about* **90 minutes** *on it.*

The marks for each question are given beside the question.

Question 1

A curve $\alpha : \mathbf{R} \longrightarrow \mathbf{E}^3$ is defined by

$$\alpha(t) = (t\cosh t, t\sinh t, t^3 + 1), \quad t \in \mathbf{R}.$$

Find the velocity of α and hence find the speed when $t = 0$. [2]

Question 2

The function $f : \mathbf{E}^3 \longrightarrow \mathbf{R}$ and the vector field V on $\mathbf{E}^3$ are defined by

$$f = x^2 - 2xy + z^2 \quad \text{and} \quad V = yU_1 + xU_2 - zU_3.$$

(a) Write down df in terms of dx, dy and dz and **hence** find $V[f]$. [3]

(b) Calculate the covariant derivative $\nabla_V V$. [3]

Question 3

The 1-form ϕ and the 2-form η are defined by

$$\phi = y\,dx - z\,dy + x^2\,dz \quad \text{and} \quad \eta = x\,dx\,dy + y\,dx\,dz.$$

(a) Calculate $d\phi$. [2]

(b) Evaluate $\eta(\mathbf{v_p}, \mathbf{w_p})$, where

$$\mathbf{v_p} = (3, 1, -2)_{(1,2,1)} \quad \text{and} \quad \mathbf{w_p} = (-1, 3, 2)_{(1,2,1)}.$$ [2]

(c) Calculate $\phi \wedge \eta$, simplifying your answer as far as possible. [2]

Question 4

The functions $F : \mathbf{E}^2 \longrightarrow \mathbf{E}^3$ and $G : \mathbf{E}^3 \longrightarrow \mathbf{E}^3$ are defined by

$$F = (x^2, y^2, xy) \quad \text{and} \quad G = (xy, y, z^2).$$

(a) Write down the Jacobian matrices for F_* and G_* and evaluate G_* at $F(x, y, z)$. [3]

(b) Use the Composite Rule to find the Jacobian matrix for $(G \circ F)_*$. [2]

(c) Is G regular everywhere? If so, explain briefly why; if not, find the points at which G is not regular. [2]

Question 5

Let α be a unit-speed curve in $\mathbf{E}^3$ with non-vanishing acceleration.

(a) Define the Frenet apparatus T, κ, N, B and τ of α. Define the terms in the order given here. At each stage you may make use of any previously defined terms. [2]

(b) Write down the Frenet formulas for α. [1]

Let α be the curve in $\mathbf{E}^3$ defined by

$$\alpha(t) = \left(\sin(t/\sqrt{2}), \cos(t/\sqrt{2}), t\sqrt{2}\right), \quad t \in \mathbf{R}.$$

(c) Show that α is unit speed and find the Frenet apparatus of α. [5]

Question 6

A unit-speed curve β in $\mathbf{E}^3$ has Frenet apparatus T, κ, N, B and τ. An orthogonal transformation from $\mathbf{E}^3$ to itself is defined by the matrix

$$C = \begin{pmatrix} 0 & \frac{4}{5} & \frac{3}{5} \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 1 & 0 & 0 \end{pmatrix}.$$

Find the curvature and torsion of the curve $\gamma = C(\beta)$, explaining your answers briefly. [3]

Question 7

A surface M is parametrized by the mapping

$$\mathbf{x}(u,v) = (u+v, u-v, u^2-v^2), \quad u,v \in \mathbf{R}.$$

(a) Find the partial velocities for this parametrization. [2]

(b) Show that the point $\mathbf{p} = (1,3,3)$ lies in M and that the tangent vector

$$\mathbf{v}_\mathbf{p} = (5,1,16)_{(1,3,3)}$$

is tangent to M at $\mathbf{p}$. [3]

A mapping $F : M \longrightarrow M$ is defined by $F(\mathbf{x}(u,v)) = \mathbf{x}(2v, 3u)$.

(c) Find $F_*(\mathbf{x}_u(u,v))$ and $F_*(\mathbf{x}_v(u,v))$. Give your answers in terms of $\mathbf{x}_u(u_1,v_1)$ and $\mathbf{x}_v(u_1,v_1)$ at a suitable point $\mathbf{x}(u_1,v_1)$ of M. Give u_1 and v_1 explicitly in terms of u and v. [3]

The 1-form ϕ is defined on M as follows.

$$\phi(\mathbf{x}_u(u,v)) = u, \quad \phi(\mathbf{x}_v(u,v)) = v.$$

(d) Evaluate $(F^*\phi)(\mathbf{x}_u(u,v))$ and $(F^*\phi)(\mathbf{x}_v(u,v))$, giving your answers in terms of u and v. Hence evaluate $(F^*\phi)(\mathbf{v}_\mathbf{p})$, where $\mathbf{v}_\mathbf{p}$ is as defined in part (b). [3]

Question 8

A surface M is parametrized by the mapping

$$\mathbf{x}(u,v) = (\tan u \cos v, \tan u \sin v, v), \quad u,v \in \mathbf{R},\ 0 < u < \pi/2.$$

(a) For this parametrization, calculate the following.

(i) E, F and G. [2]

(ii) A unit normal vector field U on M. [2]

(iii) l, m and n. [2]

(iv) The Gaussian curvature function K and the mean curvature function H of M. [2]

(v) The principal curvatures k_1 and k_2 of M. [2]

(b) Does M possess any umbilic points? Justify your answer briefy by reference to your results from part (a) of this question. [2]

PART II

You may attempt not more than **THREE** *questions from this part and are advised to spend about* **80 minutes** *on it.*

Each question carries 15% of the marks for the whole examination and an indication of the allocation of marks within each question is given beside the question.

Question 9

Let α be a curve in $\mathbf{E}^3$, not necessarily unit speed, with non-vanishing acceleration and Frenet apparatus

$$v, T, \kappa, N, B, \tau.$$

(a) By differentiating $\alpha' = vT$, or otherwise, show that

(i) $\alpha' \times \alpha'' = (\kappa v^3)B$, [2]

(ii) $\alpha' \times \alpha'' \cdot \alpha''' = \tau(\kappa v^3)^2$. [3]

The elliptical helix α is defined by

$$\alpha(t) = (2\cos t, \sin t, t).$$

(b) Find the Frenet apparatus

$$v(0),\ T(0),\ \kappa(0),\ N(0),\ B(0),\ \tau(0)$$

of α at $t = 0$. [6]

The regular curve β has Frenet apparatus

$$v_\beta,\ T_\beta,\ \kappa_\beta,\ N_\beta,\ B_\beta,\ \tau_\beta,$$

where

$$\begin{aligned} \beta(0) &= (0,0,0), \\ T_\beta(0) &= \tfrac{1}{3}(2,-1,2), \\ N_\beta(0) &= \tfrac{1}{3}(1,-2,-2), \\ B_\beta(0) &= \tfrac{1}{3}(2,2,-1), \\ v_\beta = v, \quad &\kappa_\beta = \kappa, \quad \tau_\beta = -\tau. \end{aligned}$$

(c) Find an isometry F, such that $\beta = F(\alpha)$. You should express your answer in '$F = TC$' form. [4]

Question 10

The torus T is parametrized by the mapping

$$\mathbf{x}(u,v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u), \quad R > r > 0.$$

(a) Find the partial velocities and normalize them to obtain a tangent frame field E_1, E_2 on T. [3]

(b) Use E_1 and E_2 to construct a unit normal vector field, U, on T. [1]

Let S be the shape operator derived from U.

(c) Find $S(E_1)$ and $S(E_2)$ and hence write down the matrix representing S with respect to E_1 and E_2. [5]

(d) Write down the Gaussian curvature, K, the mean curvature, H, and the principal curvatures, k_1, k_2, at the point $\mathbf{x}(u,v)$ of T. [2]

(e) Explain why asymptotic directions exist at $\mathbf{x}(u,v)$ only if $\cos u < 0$. Express the asymptotic directions at $\mathbf{x}(\pi, 0)$ in terms of E_1 and E_2 at this point. [4]

Question 11

The unit-speed curve α has Frenet apparatus T, $\kappa > 0$, N, B and τ. The surface M is defined by the single patch

$$\mathbf{x}(u,v) = \alpha(u) + vN(u),$$

where you may assume that u and v are restricted to a domain that ensures that $\mathbf{x}$ is a proper patch.

(a) Find a unit normal vector field on M, expressing your answer in terms of the Frenet apparatus of α and the parameters u and v. [4]

(b) By calculating the functions E, F, G, l, m and n for M, or otherwise, show that the Gaussian curvature is non-positive everywhere on M. [6]

(c) For the special case in which α is a **circular** helix, show that M is minimal. [5]

Question 12

Let M be the helicoid parametrized by the mapping

$$\mathbf{x}(u,v) = (u\cos v, u\sin v, v)$$

and let N be the unit sphere parametrized by the mapping

$$\mathbf{y}(u,v) = (\cos u\cos v, \cos u\sin v, \sin u).$$

The mapping $G : M \longrightarrow N$ is defined by

$$G(\mathbf{x}(u,v)) = \mathbf{y}(\tan^{-1} u, v - \pi/2).$$

(a) Show that

$$U(u,v) = \frac{1}{(1+u^2)^{1/2}}(\sin v, -\cos v, u)$$

defines a unit normal vector field on M. [3]

(b) Explain why $\mathbf{y}(u,v)$ may be taken as the vector part of a unit normal vector field V on N. [1]

(c) Calculate

$$G_*(\mathbf{x}_u(u,v)) \quad \text{and} \quad G_*(\mathbf{x}_v(u,v)),$$

expressing your answers as linear combinations of $\mathbf{y}_u(u_1,v_1)$ and $\mathbf{y}_v(u_1,v_1)$ for suitable values of u_1 and v_1. [3]

The area 2-form η on M is defined by

$$\eta(\mathbf{v_p}, \mathbf{w_p}) = \mathbf{v_p} \times \mathbf{w_p} \cdot U(u,v),$$

where $\mathbf{p} = \mathbf{x}(u,v)$.

The area 2-form ζ on N is defined by

$$\zeta(\mathbf{v_p}, \mathbf{w_p}) = \mathbf{v_p} \times \mathbf{w_p} \cdot V(u,v),$$

where $\mathbf{p} = \mathbf{y}(u,v)$.

(d) By evaluating both $G^*\zeta$ and η on the pair

$$(\mathbf{x}_u(u,v), \mathbf{x}_v(u,v)),$$

or otherwise, show that

$$G^*\zeta = \frac{-1}{(1+u^2)^2}\eta.$$

[7]

(e) Write down the Gaussian curvature function K of M. [1]

Question 13

A surface M is covered by a single orthogonal patch $\mathbf{x}$ with the property that

$$E = \frac{u^2}{(1+u^2)^2}, \quad F = 0 \quad \text{and} \quad G = \frac{1}{(1+u^2)^2}.$$

The domain of $\mathbf{x}$ is restricted to $u > 0$.

(a) Write down a tangent frame field E_1, E_2 on M, giving your answers in terms of $\mathbf{x}_u$, $\mathbf{x}_v$, E and G. [2]

(b) Show, by direct calculation that the 1-forms

$$\theta_1 = \frac{u}{1+u^2}du \quad \text{and} \quad \theta_2 = \frac{1}{1+u^2}dv$$

are dual to E_1 and E_2. [3]

(c) Use the first structural equations

$$d\theta_1 = \omega_{12} \wedge \theta_2 \quad \text{and} \quad d\theta_2 = -\omega_{12} \wedge \theta_1$$

to find ω_{12} in terms of θ_1 and θ_2. [7]

(d) Use the second structural equation

$$d\omega_{12} = -K\theta_1 \wedge \theta_2$$

to show that the Gaussian curvature of M is constant. [3]

[END OF QUESTION PAPER]

Printed in the United Kingdom by Hobbs the Printers Ltd,
Second Avenue, Southampton, SO9 2UZ

M434 Solutions to Specimen Examination Paper

The solutions that follow are, in some cases, slightly longer than is strictly necessary for completeness. Where this is the case the comments indicate possible places where the solution might be shortened.

As a general rule, minor slips or errors are penalized $\frac{1}{2}$ mark and then followed through. If an error makes the rest of a question *substantially* easier, then the rest will be followed through for a maximum of about half the available marks.

PART I

Question 1

We have

$$\alpha'(t) = (\cosh t + t\sinh t, \sinh t + t\cosh t, 3t^2)_{\alpha(t)}.$$

1 The velocity is a tangent vector, so the point of application is needed.

At $t = 0$, $\alpha'(0) = (1, 0, 0)_{(0,0,1)}$, so the speed is

$$\|\alpha'(0)\| = \sqrt{1+0+0} = 1.$$

1 The mark would be given if the strategy is clear from the working.

Question 2

(a) We have

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ &= (2x-2y)\,dx + (-2x)\,dy + (2z)\,dz \\ &= (2x-2y)\,dx - 2x\,dy + 2z\,dz. \end{aligned}$$

$\frac{1}{2}$,1 First half mark is for method, second for result. The first line could be omitted, providing that the structure is shown clearly, as in the second line.

Now we use $V[f] = df(V)$.

$$\begin{aligned} V[f] &= ((2x-2y)\,dx - 2x\,dy + 2z\,dz)(yU_1 + xU_2 - zU_3) \\ &= (2x-2y)\,dx(yU_1 + xU_2 - zU_3) - 2x\,dy(yU_1 + xU_2 - zU_3) \\ &\quad + 2z\,dz(yU_1 + xU_2 - zU_3) \quad \text{(linearity)} \\ &= (2x-2y)(y+0+0) - 2x(0+x-0) + 2z(0+0-z) \\ &\quad \text{(linearity and } dx_i(U_j) = \delta_{ij}) \\ &= 2xy - 2y^2 - 2x^2 - 2z^2 = 2(xy - y^2 - x^2 - z^2). \end{aligned}$$

1,$\frac{1}{2}$ Linearity and the action of dx etc. on the U_i should be clearly visible in the answer, even if not explicitly commented on.

(b) We use linearity and the fact that differentiation with respect to the natural frame field corresponds to partial differentiation.

$$\begin{aligned} \nabla_V V &= \nabla_{(yU_1+xU_2-zU_3)}(yU_1 + xU_2 - zU_3) \\ &= y\nabla_{U_1}(yU_1 + xU_2 - zU_3) + x\nabla_{U_2}(yU_1 + xU_2 - zU_3) \\ &\quad - z\nabla_{U_3}(yU_1 + xU_2 - zU_3) \quad \text{(linearity)} \\ &= y(0 + U_2 + 0) + x(U_1 + 0 + 0) - z(0 + 0 - U_3) \\ &= yU_2 + xU_1 + zU_3 = xU_1 + yU_2 + zU_3. \end{aligned}$$

1,1,1 One each for method, intermediate steps and final result.

Question 3

(a) We find the differentials of the coefficient functions.

$$\begin{aligned} d\phi &= d(y) \wedge dx - d(z) \wedge dy + d(x^2) \wedge dz \\ &= dy\, dx - dz\, dy + 2x\, dx\, dz \\ &= -dx\, dy + dy\, dz + 2x\, dx\, dz \quad \text{(alternation).} \end{aligned}$$

1,1 Marks for method and final result. The method marks would be given provided the working is clear.

(b) We apply linearity and

$$dx\, dy(\mathbf{v_p}, \mathbf{w_p}) = dx(\mathbf{v_p})dy(\mathbf{w_p}) - dx(\mathbf{w_p})dy(\mathbf{v_p}),$$

etc. Thus

$$\begin{aligned} \eta(\mathbf{v_p}, \mathbf{w_p}) &= (x\, dx\, dy + y\, dx\, dz)(\mathbf{v_p}, \mathbf{w_p}) \\ &= x(\mathbf{p})dx\, dy(\mathbf{v_p}, \mathbf{w_p}) + y(\mathbf{p})dx\, dz(\mathbf{v_p}, \mathbf{w_p}) \\ &= x(\mathbf{p})(dx(\mathbf{v_p})dy(\mathbf{w_p}) - dx(\mathbf{w_p})dy(\mathbf{v_p})) \\ &\qquad + y(\mathbf{p})(dx(\mathbf{v_p})dz(\mathbf{w_p}) - dx(\mathbf{w_p})dz(\mathbf{v_p})) \\ &= 1(3 \cdot 3 - (-1) \cdot 1) + 2(3 \cdot 2 - (-1) \cdot (-2)) \\ &= 9 + 1 + 2(6 - 2) \\ &= 18. \end{aligned}$$

2 One and a half for clearly visible method and correct working, last half for the correct answer.

(c) The technique uses 'repeats are zero' and the alternation rule.

$$\begin{aligned} \phi \wedge \eta &= (y\, dx - z\, dy + x^2\, dz) \wedge (x\, dx\, dy + y\, dx\, dz)) \\ &= yx\, dx\, dx\, dy - zx\, dy\, dx\, dy + x^3\, dz\, dx\, dy \\ &\qquad + y^2\, dx\, dx\, dz - zy\, dy\, dx\, dz + x^2y\, dz\, dx\, dz \\ &= 0 - 0 + x^3\, dz\, dx\, dy + 0 - zy\, dy\, dx\, dz + 0 \\ &= -x^3\, dx\, dz\, dy + zy\, dx\, dy\, dz \\ &= x^3\, dx\, dy\, dz + zy\, dx\, dy\, dz \\ &= (x^3 + zy)dx\, dy\, dz. \end{aligned}$$

2 Same distribution as for previous part.

Question 4

(a) The Jacobian matrices are

$$F_* = \begin{pmatrix} 2x & 0 \\ 0 & 2y \\ y & x \end{pmatrix} \quad \text{and} \quad G_* = \begin{pmatrix} y & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2z \end{pmatrix}.$$

Substituting x^2 for x etc. in G_*, we get

$$G_*(F) = \begin{pmatrix} y^2 & x^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2xy \end{pmatrix}.$$

2,1 Main point to watch for is not writing down the transpose of the required matrices.

(b) We have

$$\begin{aligned} (G \circ F)_* = G_*(F)\, F_* &= \begin{pmatrix} y^2 & x^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2xy \end{pmatrix} \begin{pmatrix} 2x & 0 \\ 0 & 2y \\ y & x \end{pmatrix} \\ &= \begin{pmatrix} 2xy^2 & 2x^2y \\ 0 & 2y \\ 2xy^2 & 2x^2y \end{pmatrix}. \end{aligned}$$

1,1 1 for clear application of the Composite rule, 1 for accurate working and answer.

(c) Regularity requires G_* to be one–one. Since the matrix is square, we can use the determinant.

$$\det(G_*) = y(2z - 0) = 2yz.$$

Thus, G is *not* regular everywhere. It fails to be regular where $y = 0$ (the xz-plane) and where $z = 0$ (the xy-plane).

$\frac{1}{2}$,$1\frac{1}{2}$ First half is for a bald answer. Rest for justification and working.

Question 5

(a) One of the few items from the course that needs learning.

$$T = \alpha',\ \kappa = \|T'\|,\ N = T'/\kappa,\ B = T \times N,\ \tau = -B' \cdot N.$$ 2 — $\frac{1}{2}$ off for each error down to a minimum of 0.

(b) Frenet formulas are:

$$\begin{aligned} T' &= \kappa N, \\ N' &= -\kappa T + \tau B, \\ B' &= -\tau N. \end{aligned}$$

1 — Marking as in previous part. Matrix form of the equations is equally acceptable.

(c) We have

$$\alpha'(t) = \Big((1/\sqrt{2})\cos(t/\sqrt{2}), -(1/\sqrt{2})\sin(t/\sqrt{2}), 1/\sqrt{2}\Big)_{\alpha(t)}.$$

Hence

$$\|\alpha'(t)\|^2 = (1/2)(\cos^2 t + \sin^2 t) + (1/2) = 1.$$

Thus α is unit speed.

$$\begin{aligned} T = \alpha'(t) &= \Big((1/\sqrt{2})\cos(t/\sqrt{2}), -(1/\sqrt{2})\sin(t/\sqrt{2}), 1/\sqrt{2}\Big); \\ T' &= \Big(-(1/2)\sin(t/\sqrt{2}), -(1/2)\cos(t/\sqrt{2}), 0\Big), \\ \kappa &= \|T'\| = 1/2; \\ N = T'/\kappa &= \Big(-\sin(t/\sqrt{2}), -\cos(t/\sqrt{2}), 0\Big); \\ B = T \times N &= \Big((1/\sqrt{2})\cos(t/\sqrt{2}), -(1/\sqrt{2})\sin(t/\sqrt{2}), -(1/\sqrt{2})\Big); \\ B' &= \Big(-(1/2)\sin(t/\sqrt{2}), -(1/2)\cos(t/\sqrt{2}), 0\Big), \\ \tau = -B' \cdot N &= -((1/2)(\sin^2 t + \cos^2 t)) \\ &= -(1/2). \end{aligned}$$

$10 \times \frac{1}{2}$ — There are 10 important steps in the calculations. The above is quite sufficient, since the use of Frenet is clear from the layout.

Question 6

An orthogonal matrix defines an isometry. $\frac{1}{2}$ — Although a short question, it does use one of the fundamental results about curves.

Hence, the curvature of γ is the same as the curvature of α, that is κ.

Also 1

$$\text{Torsion of } \gamma = \operatorname{sgn} C \times \tau.$$

We can find $\operatorname{sgn} C$ from the determinant. $\frac{1}{2}$

$$\det(C) = -(4/5)(3/5 - 0) + (3/5)(0 - (-4/5)) = -1,$$ $\frac{1}{2}$

so the torsion of γ is $-\tau$. $\frac{1}{2}$

Question 7

(a) We have

$$\mathbf{x}_u(u,v) = (1,1,2u),$$
$$\mathbf{x}_v(u,v) = (1,-1,-2v).$$

1,1 1 each, $\frac{1}{2}$ off each error, as usual. Omitting the parameters on the left hand sides is *just* allowable here but very unwise in the rest of the question.

(b) We require

$$\mathbf{x}(u,v) = (u+v, u-v, u^2+v^2) = (1,3,3).$$

The only possibility is $u+v=1$ and $u-v=3$. Thus $u=2$, $v=-1$ and

$$\mathbf{p} = \mathbf{x}(2,-1) \in M.$$

$1\frac{1}{2}$ $\frac{1}{2}$ each for method and each parameter.

At $\mathbf{p}$

$$\mathbf{x}_u(2,-1) = (1,1,4),$$
$$\mathbf{x}_v(2,-1) = (1,-1,2).$$

Since $\mathbf{v_p} = 3\mathbf{x}_u(2,-1) + 2\mathbf{x}_v(2,-1)$ it is tangent to M at $\mathbf{p}$.

$1\frac{1}{2}$ The answer should show clearly that 'tangent to M at $\mathbf{p}$' means 'a linear combination of the partial velocities at $\mathbf{p}$'.
The explicit definitions of u_1 and v_1 must appear somewhere.

(c) Apply the chain rule. Let $u_1 = 2v$, $v_1 = 3u$.

$$\begin{aligned}
F_*(\mathbf{x}_u(u,v)) &= \frac{\partial}{\partial u}F(\mathbf{x}(u,v)) \\
&= \frac{\partial}{\partial u}F(2v,3u) \\
&= \mathbf{x}_u(2v,3u)\frac{\partial(2v)}{\partial u} + \mathbf{x}_v(2v,3u)\frac{\partial(3u)}{\partial u} \\
&= 0 + 3\mathbf{x}_v(u_1,v_1).
\end{aligned}$$

$$\begin{aligned}
F_*(\mathbf{x}_v(u,v)) &= \frac{\partial}{\partial v}F(\mathbf{x}(u,v)) \\
&= \frac{\partial}{\partial v}F(2v,3u) \\
&= \mathbf{x}_u(2v,3u)\frac{\partial(2v)}{\partial v} + \mathbf{x}_v(2v,3u)\frac{\partial(3u)}{\partial v} \\
&= 2\mathbf{x}_u(u_1,v_1) + 0.
\end{aligned}$$

3 $\frac{1}{2}$ each for new parameters, explicit use of chain rule (make sure that the form of your answer shows that you know it), each calculation and each result.

(d) Apply the definition.

$$\begin{aligned}
(F^*\phi)(\mathbf{x}_u(u,v)) &= \phi(F_*(\mathbf{x}_u(u,v))) \\
&= \phi(3\mathbf{x}_v(u_1,v_1)) \quad \text{(from above)} \\
&= 3\phi(\mathbf{x}_v(u_1,v_1)) \quad \text{(linearity)} \\
&= 3v_1 = 9u.
\end{aligned}$$

$\frac{1}{2}$

$\frac{1}{2}$ Either the reasons should appear explicitly or the layout should make them clear.

Similarly

$$\begin{aligned}
(F^*\phi)(\mathbf{x}_v(u,v)) &= \phi(F_*(\mathbf{x}_v(u,v))) \\
&= \phi(2\mathbf{x}_u(u_1,v_1)) \quad \text{(from above)} \\
&= 2\phi(\mathbf{x}_u(u_1,v_1)) \quad \text{(linearity)} \\
&= 2u_1 = 4v.
\end{aligned}$$

$\frac{1}{2}$ Provided 'similarly' appears, this could be much briefer.

Using linearity,

$$\begin{aligned}
(F^*\phi)(\mathbf{v_p}) &= (F^*\phi)(3\mathbf{x}_u(2,-1) + 2\mathbf{x}_v(2,-1)) \\
&= 3(F^*\phi)(\mathbf{x}_u(2,-1)) + 2(F^*\phi)(\mathbf{x}_v(2,-1)) \\
&= 3 \times 9 \times 2 + 2 \times 4 \times (-1) \\
&= 54 - 8 = 46.
\end{aligned}$$

$1\frac{1}{2}$ $\frac{1}{2}$ for use of linearity, correct application of ϕ and the result.

Question 8

(a) (i) First, the partial velocities.

$$\mathbf{x}_u(u,v) = (\sec^2 u \cos v, \sec^2 u \sin v, 0)$$
$$= \sec^2 u(\cos v, \sin v, 0),$$
$$\mathbf{x}_u(u,v) = (-\tan u \sin v, \tan u \cos v, 1).$$

Here the parameters can safely be omitted as they are the same throughout.

Hence

$$E = \mathbf{x}_u \cdot \mathbf{x}_u$$
$$= \sec^4 u(\sin^2 v + \cos^2 v) = \sec^4 u,$$
$$F = \mathbf{x}_u \cdot \mathbf{x}_v$$
$$= \sec u \tan u(-\sin v \cos v + \cos v \sin v) + 0 = 0,$$
$$G = \mathbf{x}_v \cdot \mathbf{x}_v$$
$$= \tan^2 u(\sin^2 v + \cos^2 v) + 1$$
$$= \tan^2 u + 1 = \sec^2 u.$$

2 **$\frac{1}{2}$ off for each error.**

(ii) Use the partial velocities to construct U.

$$U = \frac{\mathbf{x}_u \times \mathbf{x}_v}{||\mathbf{x}_u \times \mathbf{x}_v||}$$
$$= \frac{\sec^2 u(\sin v, -\cos v, \tan u)}{||\mathbf{x}_u \times \mathbf{x}_v||}$$
$$= \frac{\sec^2 u(\sin v, -\cos v, \tan u)}{\sec^2 u\sqrt{1+\tan^2 u}}$$
$$= \frac{1}{\sec u}(\sin v, -\cos v, \tan u).$$

2 **As above.**

(iii) We now need the second derivatives.

$$\mathbf{x}_{uu} = (2\sec u \sec u \tan u \cos v, 2\sec u \sec u \tan u \sin v,)$$
$$= 2\sec^2 u \tan u(\cos v, \sin v, 0),$$
$$\mathbf{x}_{uv} = (-\sec^2 u \sin v, \sec^2 u \cos v, 0)$$
$$= \sec^2 u(-\sin v, \cos v, 0),$$
$$\mathbf{x}_{vv} = (-\tan u \cos v, -\tan u \sin v, 0)$$
$$= -\tan u(\cos v, \sin v, 0).$$

Hence

$$l = U \cdot \mathbf{x}_{uu} = 0,$$
$$m = U \cdot \mathbf{x}_{uv} = \sec u(-\sin^2 v - \cos^2 v) = -\sec u,$$
$$n = U \cdot \mathbf{x}_{vv} = 0.$$

2 **1 for general method, 1 for results.**

(iv) Use the formulas involving the functions found above.

$$K = \frac{ln - m^2}{EG - F^2} = \frac{0 - (-\sec u)^2}{\sec^6 u - 0} = -\frac{1}{\sec^4 u}.$$
$$H = \frac{Gl + En - 2Fm}{2(EG - F^2)} = \frac{0+0-0}{2\sec^6 u} = 0.$$

2 **$\frac{1}{2}$ each for applying formulas and for results.**

(v) The principal curvatures k_1 and k_2 are solutions of

$$k^2 - 2Hk + K = 0,$$

that is

$$k^2 - \frac{1}{\sec^4 u} = 0.$$

Thus, $k_1 = 1/\sec^2 u$ and $k_2 = -1/\sec^2 u$.

2 **1 each method and results.**

(b) At an umbilic point, $k_1 = k_2$. Since $K = k_1 k_2 < 0$ in this case, there can be no umbilic point. Note that, although $K = -\cos^4 u$, the range given for u prevents $K = 0$ happening.

2 **1 each for use of $k_1 = k_2$ and deductions from it.**

Comment

Both above and in PART II below much of the commentary could well be omitted **provided** that the layout and form of your calculations make it clear what you are doing. For example, you should not omit zero terms initially in a calculation based on the 2-variable chain rule.

If time permits, a few words do help the readability of a solution, even in the case of an entirely computational question.

PART II

Question 9

(a) When the phrase 'by ...or otherwise' appears it is usually wise to do the question by the suggested method!

(i) We have, differentiating $\alpha' = vT$,

$$\begin{aligned}\alpha'' &= v'T + vT' \quad \text{(Leibniz)}\\ &= v'T + v(\kappa vN) \quad \text{(Frenet)}.\end{aligned}$$

Hence

$$\begin{aligned}\alpha' \times \alpha'' &= vT \times (v'T + \kappa v^2 N)\\ &= v'T \times T + \kappa v^2 T \times N\\ &= 0 + \kappa v^3 B.\end{aligned}$$

2 **$\frac{1}{2}$ each basic step. The commonest error is to forget that v may well not be constant, thus the term with v' must be shown.**

(ii) Differentiating again

$$\begin{aligned}\alpha''' &= v''T + v'T' + (\kappa v^2)'N + (\kappa v^2)N'\\ &= v'' + v'(\kappa vN) + (\kappa v^2)'N + (\kappa v^2)(-\kappa vT + \tau vB).\end{aligned}$$

Since we require the dot product of this with $\alpha' \times \alpha''$, which is a multiple of B, there is no point in further simplification.

$$\begin{aligned}\alpha' \times \alpha'' \cdot \alpha'' &= 0 + 0 + 0 + (\kappa v^3)(\kappa v^2)(\tau v)\\ &= \tau(\kappa v^3)^2.\end{aligned}$$

3 **Errors are penalized $\frac{1}{2}$ and followed through.**

(b) We can apply the results above to α. We only need the values of the derivatives at $t = 0$

$$\begin{aligned}\alpha'(t) &= (-2\sin t, \cos t, 1),\\ \alpha''(t) &= (-2\cos t, -\sin t, 0),\\ \alpha'''(t) &= (2\sin t, -\cos t, 0).\end{aligned}$$

Hence

$$\begin{aligned}\alpha'(0) &= (0, 1, 1),\\ \alpha''(0) &= (-2, 0, 0),\\ \alpha'''(0) &= (0, -1, 0).\end{aligned}$$

So

$$\begin{aligned}(\alpha' \times \alpha'')(0) &= (0, -2, 2),\\ (\alpha' \times \alpha'' \cdot \alpha''')(0) &= 2.\end{aligned}$$

Thus

$$\begin{aligned}v(0) &= \|\alpha'(0)\| = \sqrt{2},\\ T(0) &= \frac{\alpha'(0)}{v(0)} = \frac{1}{\sqrt{2}}(0, 1, 1),\\ (\kappa v^3 B)(0) &= (0, -2, 2),\\ (\tau(\kappa v^3)^2)(0) &= 2.\end{aligned}$$

Taking the modulus of the third equation gives

$$(\kappa v^3)(0) = 2\sqrt{2},$$

and so

$$\kappa(0) = \frac{2\sqrt{2}}{v^3(0)} = \frac{2\sqrt{2}}{2\sqrt{2}} = 1.$$

Hence
$$B(0) = \frac{1}{2\sqrt{2}}(0,-2,2) = \frac{1}{\sqrt{2}}(0,-1,1).$$
Next we calculate N from $B \times T = N$.
$$\begin{aligned} N(0) &= B(0) \times T(0) \\ &= \tfrac{1}{2}(-2,0,0) \\ &= (-1,0,0). \end{aligned}$$
Finally, we have $\tau(\kappa v^3)^2 = 2$, at $t = 0$, so
$$\begin{aligned} \tau(0) &= \frac{2}{((\kappa v^3)^2)(0)} \\ &= \frac{2}{(2\sqrt{2})^2} = \frac{2}{8} = \frac{1}{4}. \end{aligned}$$

6 There are 12 essential steps in the above, scoring $\frac{1}{2}$ each.

(c) The given information is enough to ensure that the requested isometry *does* exist. Because the sign of τ is reversed in β, we must map $B(0)$ to $-B_\beta(0)$. Thus we must map the frame with attitude matrix
$$A = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$
to the frame with attitude matrix
$$B = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$
Using the formula $C = {}^tBA$, or working from first principles, we have
$$\begin{aligned} C &= {}^tBA \\ &= \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 \\ \frac{2}{3} & \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{3} & \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \end{aligned}$$
Finally, we must find T. Since
$$C(\alpha(0)) = \left(-\tfrac{2}{3}, \tfrac{4}{3}, \tfrac{4}{3}\right),$$
and the translation must take this to $\beta(0) = (0,0,0)$, we have
$$T = \left(x + \tfrac{2}{3}, y - \tfrac{4}{3}, z - \tfrac{4}{3}\right).$$

3,1 Get the signs correct for C and be sure to use $C(\alpha(0))$, not $\alpha(0)$ in the calculation of T.

Question 10

(a) We have

$$\begin{aligned}\mathbf{x}_u &= (-r\sin u\cos v, -r\sin u\sin v, r\cos u)\\ &= r(-\sin u\cos v, -\sin u\sin v, \cos u),\\ \mathbf{x}_v &= (-(R+r\cos u)\sin v, (R+r\cos u)\cos v, 0)\\ &= (R+r\cos u)(-\sin v, \cos v, 0).\end{aligned}$$

Since $||\mathbf{x}_u|| = r$ and $||\mathbf{x}_v|| = (R + r\cos u)$, we have

$$E_1 = (-\sin u\cos v, -\sin u\sin v, \cos u) \quad \text{and} \quad E_2 = (-\sin v, \cos v, 0).$$

We have normalized but also need to check $E_1 \cdot E_2 = 0.1$ for partial velocities, 3
$1\frac{1}{2}$ for normalizing, $\frac{1}{2}$ for checking orthogonality.

$$E_1 \cdot E_2 = \sin u\cos v\sin v - \sin u\sin v\cos v = 0.$$

(b) Since E_1 and E_2 are a tangent frame field, we can take

$$\begin{aligned}U &= E_1 \times E_2\\ &= (-\cos u\cos v, -\cos u\sin v, -\sin u).\end{aligned}$$ 1

(c) Since it is easy to find $S(\mathbf{x}_u)$ as $-\partial U/\partial u$, etc. we work in terms of the partial velocities.

$$\begin{aligned}S(E_1) &= S\left(\frac{\mathbf{x}_u}{r}\right)\\ &= \frac{1}{r}S(\mathbf{x}_u) \quad \text{(linearity)}\\ &= -\frac{1}{r}\frac{\partial U}{\partial u}\\ &= -\frac{1}{r}(\sin u\cos v, \sin u\sin v, -\cos u)\\ &= \frac{1}{r}(-\sin u\cos v, -\sin u\sin v, \cos u) = \frac{1}{r}E_1.\end{aligned}$$

Similarly,

$$\begin{aligned}S(E_2) &= S\left(\frac{\mathbf{x}_v}{R+r\cos u}\right)\\ &= \frac{1}{R+r\cos u}S(\mathbf{x}_v) \quad \text{(linearity)}\\ &= -\frac{1}{R+r\cos u}\frac{\partial U}{\partial v}\\ &= -\frac{1}{R+r\cos u}(\cos u\sin v, -\cos u\cos v, 0)\\ &= \frac{\cos u}{R+r\cos u}(-\sin v, \cos v, 0)\\ &= \frac{\cos u}{R+r\cos u}E_2.\end{aligned}$$

It follows that the matrix is 2,2,1

$$\begin{pmatrix}\frac{1}{r} & 0\\ 0 & \frac{\cos u}{R+r\cos u}\end{pmatrix}.$$

2 of the marks are for correct application of linearity etc. 2 are for the calculations and 1 for the matrix. There is no danger of transposition here but do recall that the coefficients of the basis form the columns in the matrix.

(d) From the above, we have

$$\begin{aligned} K &= \det(S) \\ &= \frac{\cos u}{r(R + r\cos u)}. \\ H &= \tfrac{1}{2}\mathrm{trace}(S) \\ &= \frac{R + 2r\cos u}{2r(R + r\cos u)}, \\ k_1 &= \frac{1}{r}, \\ k_2 &= \frac{\cos u}{R + r\cos u}. \end{aligned}$$

2 $\frac{1}{2}$ off each error. Since E_1 and E_2 are eigenvectors of S, the principal curvatures can be read off.

(e) Suppose $\mathbf{u} = aE_1 + bE_2$ points in an asymptotic direction. Then $S(\mathbf{u}) \cdot \mathbf{u} = 0$. Now

$$\begin{aligned} S(\mathbf{u}) &= S(aE_1 + bE_2) \\ &= aS(E_1) + bS(E_2) \\ &= a\frac{1}{r}E_1 + b\frac{\cos u}{R + r\cos u}E_2. \end{aligned}$$

Hence

$$\begin{aligned} S(\mathbf{u}) \cdot \mathbf{u} &= \left(a\frac{1}{r}E_1 + b\frac{\cos u}{R + r\cos u}E_2\right) \cdot (aE_1 + bE_2) \\ &= a^2\frac{1}{r} + b^2\frac{\cos u}{R + r\cos u}. \end{aligned}$$

2 If you wanted, you could quote *O'Neill*, that asymptotic directions exist at points where $K < 0$. The end of the argument is the same.

This can only have solutions for a and b if the terms are different signs which requires $\cos u < 0$.

For $u = \pi$, we have $\mathbf{u}$ asymptotic if

$$a^2\frac{1}{r} + b^2\frac{-1}{R - r} = 0.$$

The easiest solutions to this give the asymptotic directions as

$$\sqrt{r}\,E_1 \pm \sqrt{R - r}\,E_2.$$

2 If you quoted *O'Neill* above, there would be more calculation in this part and the mark distribution would be $1 + 3$ rather than $2 + 2$.

Question 11

(a) We use the partial velocities to construct the unit normal vector field U on M.

We are using dashes to denote differentiation with respect to u.

$$\begin{aligned} \mathbf{x}_u(u, v) &= \alpha'(u) + vN'(u) \\ &= T(u) + v(-\kappa T(u) + \tau B(u)) \quad \text{(Frenet)} \\ &= (1 - v\kappa)T(u) + \tau B(u), \\ \mathbf{x}_v(u, v) &= N(u). \end{aligned}$$

$1\frac{1}{2}$ $\frac{1}{2}$

Thus, dropping the parameter u for brevity,

$$\begin{aligned} \mathbf{x}_u \times \mathbf{x}_v &= (1 - v\kappa)T \times N + \tau B \times N \\ &= (1 - v\kappa)B - \tau T \\ &= -\tau T + (1 - v\kappa)B. \end{aligned}$$

1

We have

$$\|\mathbf{x}_u \times \mathbf{x}_v\| = \left((1 - v\kappa)^2 + \tau^2\right)^{1/2},$$

$\frac{1}{2}$

so

$$U(u, v) = \frac{1}{\left((1 - v\kappa)^2 + \tau^2\right)^{1/2}}\left(-\tau T + (1 - v\kappa)B\right).$$

$\frac{1}{2}$

(b) From the last part,

$$\begin{aligned}
E &= ((1-v\kappa)T(u)+\tau B(u))\cdot((1-v\kappa)T(u)+\tau B(u)) \\
&= (1-v\kappa)^2+\tau^2, \\
F &= ((1-v\kappa)T(u)+\tau B(u))\cdot N(u) \\
&= 0, \\
G &= N(u)\cdot N(u) = 1.
\end{aligned}$$

$1\frac{1}{2}$ $\quad \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$

Next, the second derivatives. Remember that the dash denotes differentiation with respect to u.

$$\begin{aligned}
\mathbf{x}_{uu} &= (1-v\kappa)T' + (1-v\kappa)'T + \tau B' + \tau' B \\
&= (1-v\kappa)(\kappa N) + (1-v\kappa)'T + \tau(-\tau N) + \tau' B \\
&= (1-v\kappa)'T + (\kappa - v\kappa^2 - \tau^2)N + \tau' B \\
&= -v\kappa' T + (\kappa - v\kappa^2 - \tau^2)N + \tau' B. \\
\mathbf{x}_{uv} &= -\kappa T, \\
\mathbf{x}_{vv} &= 0.
\end{aligned}$$

2 $\quad \frac{1}{2}$ **off each error.**

Hence

$$\begin{aligned}
l &= \frac{-v\tau\kappa' + \tau'(1-v\kappa)}{((1-v\kappa)^2+\tau^2)^{1/2}}, \\
m &= \frac{-\kappa\tau}{((1-v\kappa)^2+\tau^2)^{1/2}}, \\
n &= 0.
\end{aligned}$$

1

We now calculate K from

$$\begin{aligned}
K &= \frac{ln - m^2}{EG - F^2} \\
&= \frac{0 - (\kappa\tau)^2 / \left((1-v\kappa)^2+\tau^2\right)}{((1-v\kappa)^2+\tau^2)} \\
&= \frac{-\kappa^2\tau^2}{((1-v\kappa)^2+\tau^2)^2}.
\end{aligned}$$

1

Since the numerator is the negative of a square, it cannot be positive. Thus K is non-positive everywhere on M.

$\frac{1}{2}$

(c) We need an expression for the mean curvature H.

$$\begin{aligned}
H &= \frac{Gl + En - 2Fm}{2\,(EG - F^2)} \\
&= \frac{(-v\tau\kappa' + \tau'(1-v\kappa)) \big/ \left(((1-v\kappa)^2+\tau^2)^{1/2}\right)}{2\,((1-v\kappa)^2+\tau^2)} \\
&= \frac{-v\tau\kappa' + \tau'(1-v\kappa)}{2\,((1-v\kappa)^2+\tau^2)^{3/2}}.
\end{aligned}$$

3

In the case of a circular helix, both κ and τ are constant, so

$$\kappa' = \tau' = 0$$

1

and hence $H = 0$ everywhere on M. That is, M is minimal.

1

Question 12

(a) The partial velocities are

$$\begin{aligned}
\mathbf{x}_u(u,v) &= (\cos v, \sin v, 0), \\
\mathbf{x}_v(u,v) &= (-u\sin v, u\cos v, 1).
\end{aligned}$$

As defined,

$$\begin{aligned}
U\cdot\mathbf{x}_u &= \frac{1}{(1+u^2)^{1/2}}(-\sin v\cos v + \sin v\cos v) = 0, \\
U\cdot\mathbf{x}_v &= \frac{1}{(1+u^2)^{1/2}}(-u\sin^2 v - u\cos^2 v + u) = 0.
\end{aligned}$$

Hence U is normal to M.

Since

$$\|(\sin v, -\cos v, u)\|^2 = \sin^2 v + \cos^2 v + u^2 = 1 + u^2,$$

U is also unit. 3 **$\frac{1}{2}$ each for the 6 main steps.**

(b) The phrasing suggests that no actual calculations are necessary.

Since N is a *unit* sphere, $\|\mathbf{y}(u, v)\| = 1$ and points along a radius, that is normal to the surface. Hence we may use $\mathbf{y}(u, v)$ as the vector part of V. 1

(c) Apply the chain rule.

$$\begin{aligned}
G_*(\mathbf{x}_u(u, v)) &= \frac{\partial}{\partial u}(G(\mathbf{x}(u, v)) \\
&= \frac{\partial}{\partial u}(\mathbf{y}(\tan^{-1} u, v - \pi/2)) \\
&= \mathbf{y}_u(\tan^{-1} u, v - \pi/2) \times \frac{1}{1 + u^2} + \mathbf{y}_v(\tan^{-1} u, v - \pi/2) \times 0 \\
&= \frac{1}{1 + u^2}\mathbf{y}_u(u_1, v_1),
\end{aligned}$$

where $u_1 = \tan^{-1} u$ and $v_1 = v - \pi/2$.

Similarly

$$\begin{aligned}
G_*(\mathbf{x}_v(u, v)) &= \frac{\partial}{\partial v}(G(\mathbf{x}(u, v))) \\
&= \frac{\partial}{\partial v}(\mathbf{y}(\tan^{-1} u, v - \pi/2)) \\
&= \mathbf{y}_u(\tan^{-1} u, v - \pi/2) \times 0 + \mathbf{y}_v(\tan^{-1} u, v - \pi/2) \times 1 \\
&= \mathbf{y}_u(u_1, v_1).
\end{aligned}$$

3 **1 each method and results.**

(d) We have

$$\begin{aligned}
G^*\zeta(\mathbf{x}_u(u, v), \mathbf{x}_v(u, v)) &= \zeta(G_*(\mathbf{x}_u(u, v)), G_*(\mathbf{x}_v(u, v))) \\
&= \zeta\left(\frac{1}{1 + u^2}\mathbf{y}_u(u_1, v_1), \mathbf{y}_u(u_1, v_1)\right) \\
&= \frac{1}{1 + u^2}\zeta(\mathbf{y}_u(u_1, v_1), \mathbf{y}_u(u_1, v_1)) \\
&= \frac{1}{1 + u^2}\mathbf{y}_u(u_1, v_1) \times \mathbf{y}_v(u_1, v_1) \cdot V(u_1, v_1).
\end{aligned}$$

Now,

$$\begin{aligned}
\mathbf{y}_u(u, v) &= (-\sin u \cos v, -\sin u \sin v, \cos u), \\
\mathbf{y}_v(u, v) &= (-\cos u \sin v, \cos u \cos v, 0), \\
\mathbf{y}_u(u, v) \times \mathbf{y}_v(u, v) &= (\cos^2 u \cos v, \cos^2 u \sin v, \cos u \sin u) \\
&= \cos u V(u, v).
\end{aligned}$$

It follows that

$$\begin{aligned}
G^*\zeta(\mathbf{x}_u(u, v), \mathbf{x}_v(u, v)) &= \frac{1}{1 + u^2}\mathbf{y}_u(u_1, v_1) \times \mathbf{y}_v(u_1, v_1) \cdot V(u_1, v_1) \\
&= \frac{1}{1 + u^2}\cos u_1 V(u_1, v_1) \cdot V(u_1, v_1) \\
&= \frac{1}{1 + u^2}\cos u_1 \quad (V \text{ is unit}) \\
&= \frac{1}{1 + u^2}\frac{1}{(1 + u^2)^{1/2}} \\
&= \frac{1}{(1 + u^2)^{3/2}}.
\end{aligned}$$

Equally,

$$\begin{aligned}\eta(\mathbf{x}_u(u,v),\mathbf{x}_v(u,v)) &= \mathbf{x}_u(u,v)\times\mathbf{x}_v(u,v)\cdot U(u,v)\\ &= (\cos v,\sin v,0)\times(-u\sin v,u\cos v,1)\cdot U(u,v)\\ &= (\sin v,-\cos v,u)\cdot U(u,v)\\ &= \frac{1}{(1+u^2)^{1/2}}(1+u^2)\\ &= (1+u^2)^{1/2}.\end{aligned}$$

6 $12\times\frac{1}{2}$ **for each of the main steps.**

Comparing results we see that

$$G^*\zeta(\mathbf{x}_u,\mathbf{x}_v) = \frac{1}{(1+u^2)^2}\eta(\mathbf{x}_u,\mathbf{x}_v).$$

$\frac{1}{2}$

Since the partial velocities are a basis for the tangent space at each point of M, the result follows.

$\frac{1}{2}$

(e) We can say directly that, because G above is the Gauss map,

$$K = \frac{1}{(1+u^2)^2}.$$

1 **No explanation was asked for.**

Question 13

(a) Since $F=\mathbf{x}_u\cdot\mathbf{x}_v=0$, the partial velocities are orthogonal, so the frame will be in the directions of the partial velocities.

Thus

$$E_1 = \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} = \frac{\mathbf{x}_u}{\sqrt{E}},$$

$$E_2 = \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|} = \frac{\mathbf{x}_v}{\sqrt{G}}.$$

2 **$\frac{1}{2}$ deducted for errors and/or the omission of the significance of $F=0$.**

(b) We must show that $\theta_i(E_j)=\delta_{ij}$.

$$\begin{aligned}\theta_1(E_1) &= \frac{u}{1+u^2}du\left(\frac{\mathbf{x}_u}{\sqrt{E}}\right)\\ &= \frac{u}{1+u^2}\frac{1}{\sqrt{E}}du(\mathbf{x}_u)\\ &= \frac{u}{1+u^2}\frac{1+u^2}{u} = 1,\end{aligned}$$

$$\begin{aligned}\theta_1(E_2) &= \frac{u}{1+u^2}du\left(\frac{\mathbf{x}_v}{\sqrt{G}}\right)\\ &= \frac{u}{1+u^2}\frac{1}{\sqrt{G}}du(\mathbf{x}_v) = 0,\end{aligned}$$

$$\begin{aligned}\theta_2(E_1) &= \frac{1}{1+u^2}dv\left(\frac{\mathbf{x}_u}{\sqrt{E}}\right)\\ &= \frac{1}{1+u^2}\frac{1}{\sqrt{E}}dv(\mathbf{x}_u) = 0,\end{aligned}$$

$$\begin{aligned}\theta_2(E_2) &= \frac{1}{1+u^2}dv\left(\frac{\mathbf{x}_v}{\sqrt{G}}\right)\\ &= \frac{1}{1+u^2}\frac{1}{\sqrt{G}}dv(\mathbf{x}_v)\\ &= \frac{1}{1+u^2}\frac{1+u^2}{1} = 1.\end{aligned}$$

3 **1 for method, $\frac{1}{2}$ each for checking each 1-form.**

(c) We assume that

$$\omega_{12} = a\,\theta_1 + b\,\theta_2.$$ 1 Method.

Now

$$d\theta_1 = \frac{\partial}{\partial u}\left(\frac{u}{1+u^2}\right) du\, du + \frac{\partial}{\partial v}\left(\frac{u}{1+u^2}\right) dv\, du$$
$$= 0 + 0 = 0.$$ 1 $\frac{1}{2}$ each method, answer.

Similarly,

$$d\theta_2 = \frac{\partial}{\partial u}\left(\frac{1}{1+u^2}\right) du\, dv + \frac{\partial}{\partial v}\left(\frac{1}{1+u^2}\right) dv\, dv$$
$$= \frac{-2u}{(1+u^2)^2} du\, dv$$
$$= \frac{-2u}{(1+u^2)^2}\frac{1+u^2}{u}\theta_1 \left(1+u^2\right)\theta_2$$
$$= -2\theta_1 \wedge \theta_2.$$ 2 Final 1 for expressing in terms of the dual 1-forms.

Substituting in the first structural equations,

$$0 = (a\,\theta_1 + b\,\theta_2) \wedge \theta_2$$
$$= a\theta_1 \wedge \theta_2.$$

Hence $a = 0$. 1 $\frac{1}{2}$ each method, result.

$$-2\theta_1 \wedge \theta_2 = -(a\,\theta_1 + b\,\theta_2) \wedge \theta_1$$
$$= b\theta_1 \wedge \theta_2.$$ 1

Hence $b = -2$ and

$$\omega_{12} = -2\theta_2.$$ 1

(d) We have

$$d\omega_{12} = -2d\theta_2$$
$$= -2(-2\theta_1 \wedge \theta_2)$$ 1
$$= 4\theta_1 \wedge \theta_2.$$ 1

Comparing this result with the second structural equation shows that $K = -4$, a constant. 1

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The Open University